Int. J. Solids Structures, 1970, Vol. 6, pp. 333 to 351. Pergamon Press. Printed in Great Britain

COMPLEX NATURAL FREQUENCIES OF VIBRATING SUBMERGED SPHEROIDAL SHELLS

SABIH HAYEK

Department of Engineering Mechanics and Ordnance Research Laboratory, The Pennsylvania State University, University Park, Pa.

and

FRANK L. DIMAGGIO

Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, N.Y.

Abstract—The differential equations governing the free axisymmetric extensional vibrations of an elastic prolate spheroidal shell submerged in an infinite acoustic medium are obtained in prolate spheroidal coordinates using Hamilton's principle.

Solutions are obtained by a perturbation technique which converges well for shells of small eccentricity.

Numerical results are presented for the fundamental mode, frequency and acoustic impedance of steel shells in water for ratios of major to minor axis up to 1·23 over a wide range of shell length to thickness ratios.

1. INTRODUCTION

THE vibration of elastic prolate spheroidal shells vibrating *in vacuo* and in acoustic media, has not been treated extensively in the literature due to the complexity of the shell equations of motion. Analytical and numerical solutions of the vibration of prolate spheroidal shells in vacuo are given in Refs. [1, 2] and [3, 4], respectively. Analytical and numerical solutions of the vibration of submerged elastic prolate spheroidal shells were presented in Refs. [5] and [6], respectively. This paper is primarily based on the work presented in Ref. [5] in a condensed form.

In this paper the perturbation technique employed by Shiraishi and DiMaggio [2] to study free, extensional non-torsional axisymmetric vibrations *in vacuo* of elastic prolate spheroidal shells is extended to the case of submergence in an infinite acoustic medium.

Numerical results are presented for the fundamental mode, frequency and acoustic impedance of steel shells in water for ratios of major to minor axes up to 1·23 over a wide range of shell length to thickness ratios.

2. FORMULATION OF THE PROBLEM

A. *Motion of shell*

Using Flammer's [7] notation the prolate spheroidal coordinate system and geometry of the structure are shown in Figs. 1 and 2. The shell is assumed to be bounded by confocal

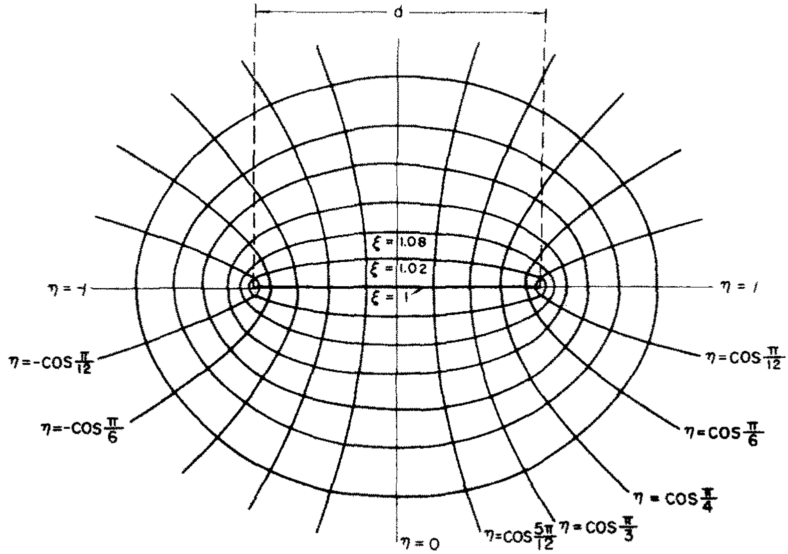

FIG. 1. Prolate spheroidal coordinate system.

spheroids defined by $\xi = a \pm h/d$, where d is the interfocal distance, h is the minimum thickness, and $\xi = a$ (eccentricity $= 1/a$) is the middle surface. The displacements w and u, respectively perpendicular and tangent to the middle surface, are termed radial and tangential.

The strain and kinetic energies of the shell, V_s and T respectively, have been obtained by Silbiger and DiMaggio [1] as

$$V_s = \frac{\pi E h}{1-\nu^2}\int_{-1}^{+1}\left\{(1-\eta^2)\left(\frac{du}{d\eta}\right)^2 + \frac{(a^2-1)a^2}{(a^2-\eta^2)^2}w^2 + \frac{2a(a^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}{a^2-\eta^2}w\frac{du}{d\eta}\right.$$

$$+\frac{\eta^2}{1-\eta^2}u^2 + \frac{a^2}{a^2-1}w^2 - \frac{2a\eta}{(a^2-1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}uw \tag{1}$$

$$\left.+2\nu\left[-\eta u\frac{du}{d\eta} + \frac{a(1-\eta^2)^{\frac{1}{2}}}{(a^2-1)^{\frac{1}{2}}}w\frac{du}{d\eta} - \frac{a(a^2-1)^{\frac{1}{2}}\eta}{(a^2-\eta^2)(1-\eta^2)^{\frac{1}{2}}}uw + \frac{a^2}{a^2-\eta^2}w^2\right]\right\}d\eta$$

$$T = \frac{\pi E h}{1-\nu^2}\frac{\rho_s}{\mu}\left(\frac{d}{2}\right)^2\frac{1-\nu}{2}\int_{-1}^{+1}(a^2-\eta^2)(\dot{u}^2+\dot{w}^2)\,d\eta \tag{2}$$

in which dots denote differentiation with respect to time, E is Young's modulus, μ is the shear modulus, ν is Poisson's ratio, ρ_s is the mass density of the shell.

The appropriate functional involving p_a, the fluid pressure on the shell surface, to be used in applying Hamilton's principle is

$$X_f = -\int p_a w\,dS \tag{3}$$

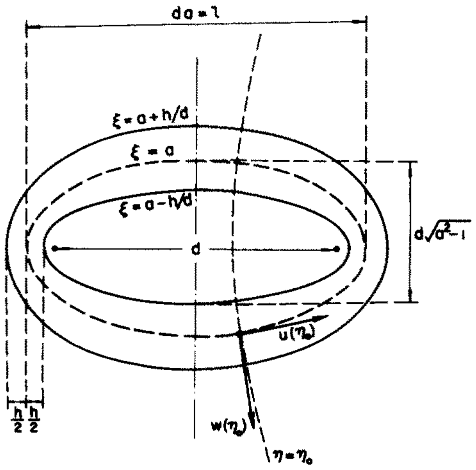

FIG. 2. Geometry of structure.

where the negative sign is due to the choice of w as positive outward. This becomes, on substituting the expression for dS, the elementary shell surface area in spheroidal coordinates, and integrating with respect to the angle of revolution

$$X_f = -\frac{\pi d^2}{2}\int_{-1}^{+1} p_a w(a^2-\eta^2)^{\frac{1}{2}}(a^2-1)^{\frac{1}{2}}\,\mathrm{d}\eta. \tag{4}$$

Application of Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T-V_s+X_f)\,\mathrm{d}t = 0 \tag{5}$$

yields the desired equations of motion:

$$L_{uu}u+L_{uw}w+\frac{\rho_s}{\mu}\left(\frac{da}{2}\right)^2 M\ddot{u} = 0 \tag{6}$$

$$L_{wu}u+L_{ww}w+\frac{\rho_s}{\mu}\left(\frac{da}{2}\right)^2 M\ddot{w} = -\frac{1}{\mu h}\left(\frac{da}{2}\right)^2\sqrt{\left(\frac{a^2-1}{a^2-\eta^2}\right)}Mp_a \tag{7}$$

where the L_{uu}, L_{uw}, L_{wu}, and L_{ww} are linear differential operators written out in Appendix I of Refs. [2] and [5], and

$$M = \frac{1-\nu}{2}\left(1-\frac{\eta^2}{a^2}\right). \tag{8}$$

Letting

$$u(\eta, t) = U(\eta)\,\mathrm{e}^{i\Omega t} \tag{9}$$

$$w(\eta, t) = W(\eta)\,\mathrm{e}^{i\Omega t} \tag{10}$$

$$p_a(\eta, t) = P_a(\eta)\,\mathrm{e}^{i\Omega t} \tag{11}$$

in which U, W, P_a and Ω are complex (with the understanding that in these equations and similar ones to follow, either the real or imaginary part of the right-hand sides may be chosen) equations (6) and (7) become

$$L_{uu}U+L_{uw}W-\lambda MU = 0 \tag{12}$$

$$L_{wu}U+L_{ww}W-\lambda MW = -\frac{1}{\mu h}\left(\frac{da}{2}\right)^2\sqrt{\left(\frac{a^2-1}{a^2-\eta^2}\right)}MP_a \tag{13}$$

where

$$\lambda = \frac{\rho_s}{\mu}\left(\frac{da}{2}\right)^2\Omega^2. \tag{14}$$

By defining matrices

$$\mathbf{L} = \begin{bmatrix} L_{uu} & L_{uw} \\ L_{wu} & L_{ww} \end{bmatrix} \tag{15}$$

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tag{16}$$

in which (although $L_{uw} \neq L_{wu}$) **L** and **M** are self adjoint, i.e. for arbitrary vectors **X** and **Y**,

$$\begin{aligned} \int_{-1}^{1} \tilde{\mathbf{X}}\mathbf{L}\mathbf{Y}\,\mathrm{d}\eta &= \int_{-1}^{1} \tilde{\mathbf{Y}}\mathbf{L}\mathbf{X}\,\mathrm{d}\eta \\ \int_{-1}^{1} \tilde{\mathbf{X}}\mathbf{M}\mathbf{Y}\,\mathrm{d}\eta &= \int_{-1}^{1} \tilde{\mathbf{Y}}\mathbf{M}\mathbf{X}\,\mathrm{d}\eta \end{aligned} \tag{17}$$

where $\tilde{\mathbf{X}}$ signifies the transpose of **X**, and

$$\mathbf{U} = \begin{bmatrix} U \\ W \end{bmatrix} \tag{18}$$

$$\mathbf{P} = \begin{bmatrix} 0 \\ -\frac{1}{\mu h}\left(\frac{da}{2}\right)^2\sqrt{\left(\frac{a^2-1}{a^2-\eta^2}\right)}P_a \end{bmatrix} = \begin{bmatrix} {}_1P \\ {}_2P \end{bmatrix} \tag{19}$$

equations (12) and (13) can be simply written as

$$(\mathbf{L}-\lambda\mathbf{M})\mathbf{U} = \mathbf{MP}. \tag{20}$$

Equation (5) also provides the natural boundary conditions

$$U(\pm 1) = 0 \tag{21}$$

$$W(\pm 1) \text{ is bounded.} \tag{22}$$

B. *Fluid potential*

If the velocity potential $\psi(\xi, \eta, t)$ is expressed as

$$\psi(\xi, \eta, t) = A\Phi(\xi, \eta)\,\mathrm{e}^{i\Omega t} \tag{23}$$

where A is an arbitrary constant, then Φ, which is complex, satisfies the scalar wave equation

$$\left[\nabla^2 + \left(\frac{\Omega}{s}\right)^2\right]\Phi = 0 \tag{24}$$

where s is the velocity of sound propagation in the fluid. With the Laplacian expressed in prolate spheroidal coordinates, the solution for the axisymmetric case is [7]

$$\Phi(c, \xi, \eta) = \sum_n S_{on}(c, \eta) R_{on}(c, \xi) = \sum_n \Phi_{on}(c, \eta, \xi) \tag{25}$$

where S_{on} and R_{on} are prolate spheroidal angle and radial functions of the first kind and

$$c = \frac{1}{2}\frac{\Omega d}{s}. \tag{26}$$

C. *Boundary conditions at shell–fluid interface*

The radial velocity of the shell is equal to that of the fluid at the surface:

$$\dot{w} = [\nabla\psi \,.\, \mathbf{e}_\xi]_{\xi=a} \tag{27}$$

where ∇ is the gradient operator, $\mathbf{e}_\xi$ is a unit vector in the radial direction, and the dot in the brackets denotes scalar multiplication. Equation (27) may be written as

$$\dot{w} = \left[\frac{1}{h_\xi}\frac{\partial\psi}{\partial_\xi}\right]_{\xi=a} \tag{28}$$

where h_ξ is the radial scale factor which, in prolate spheroidal coordinates, is

$$h_\xi = \frac{d}{2}\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}. \tag{29}$$

The pressure of the surrounding medium on the shell is given by

$$p_a = -\rho_a\left[\frac{\partial\psi}{\partial t}\right]_{\xi=a} \tag{30}$$

where ρ_a is the mass density of the fluid. From equations (10), (23), (28) and (30) one obtains

$$p_a = \rho_a\Omega^2 W\left[\frac{\Phi}{\dfrac{1}{h_\xi}\dfrac{\partial\Phi}{\partial\xi}}\right]_{\xi=a} e^{i\Omega t} \tag{31}$$

so that, from equations (11), (14), (19) and (29)

$$P_a = \rho_a\Omega^2 W\left[\frac{\Phi}{\dfrac{1}{h_\xi}\dfrac{\partial\Phi}{\partial\xi}}\right]_{\xi=a} \tag{32}$$

$${}_2P = \frac{\rho_a}{\rho_s}\frac{da/2}{h}\lambda\left\{-\frac{1}{a}\left[\frac{\Phi}{\partial\Phi/\partial\xi}\right]_{\xi=a}\right\}W. \tag{33}$$

3. EXPANSION INTO POWER SERIES IN $1/a^2$

The operator **L** of equation (15) may be expanded into

$$\mathbf{L} = \sum_{k=0}^{\infty} a^{-2k}\mathbf{L}^{(2k)} \tag{34}$$

where the elements of the self-adjoint operators

$$\mathbf{L}^{(2k)} = \begin{bmatrix} L_{uu}^{(2k)} & L_{uw}^{(2k)} \\ L_{wu}^{(2k)} & L_{ww}^{(2k)} \end{bmatrix} \tag{35}$$

are explicitly written out in Appendix I of Refs. [2] and [5] and seen to be independent of a, while the matrix **M** of equation (16) may be written as

$$\mathbf{M} = \mathbf{M}^{(0)} + \frac{1}{a^2}\mathbf{M}^{(2)} \tag{36}$$

where

$$\mathbf{M}^{(0)} = \frac{1-\nu}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{M}^{(2)} = \frac{1-\nu}{2}\begin{bmatrix} -\eta^2 & 0 \\ 0 & -\eta^2 \end{bmatrix}. \tag{37}$$

Solutions for the mode shapes **U** of equation (18), eigenvalues λ of equation (14), and pressure matrix **P** of equation (19) are sought in the form

$$\mathbf{U} = \sum_{k=0}^{\infty} a^{-2k}\mathbf{U}^{(2k)} \tag{38}$$

$$\lambda = \sum_{k=0}^{\infty} a^{-2k}\lambda^{(2k)} \tag{39}$$

$$\mathbf{P} = \sum_{k=0}^{\infty} a^{-2k}\mathbf{P}^{(2k)} \tag{40}$$

where the $\lambda^{(2k)}$ and the elements of

$$\mathbf{U}^{(2k)} = \begin{bmatrix} U^{(2k)} \\ W^{(2k)} \end{bmatrix} \tag{41}$$

and

$$\mathbf{P}^{(2k)} = \begin{bmatrix} 0 \\ {}_2P^{(2k)} \end{bmatrix} \tag{42}$$

are not functions of a.

In Appendix I it is shown that an appropriate expansion of the functions Φ_{on} of equation (25) for outgoing waves is

$$\Phi_{on}(c, \eta, \xi) = \frac{e^{-ic\xi}}{c\xi}\left\{\left[\Phi_0(\eta) + \frac{1}{\xi^2}\Phi_2(c, \eta) + \frac{1}{\xi^4}\Phi_4(c, \eta) + \dots\right] - i\left[\frac{1}{\xi}\Phi_1(c, \eta) + \frac{1}{\xi^3}\Phi_3(c, \eta) + \dots\right]\right\} \tag{43}$$

where

$$\Phi_0(\eta) = P_n(\eta) \tag{44}$$

is the Legendre polynomial of the first kind and order n. If Φ_{on} of equation (43) is substituted into equation (24) with the Laplacian operator expressed in prolate spheroidal coordinates, a recurrence equation on the Φ_j is obtained by equating to zero the coefficients of each power of $1/\xi$:

$$\begin{aligned}(-1)^j 2(j+1)c\Phi_{j+1} + \left[\frac{\mathrm{d}}{\mathrm{d}\eta}(1-\eta^2)\frac{\mathrm{d}}{\mathrm{d}\eta} + c^2(1-\eta^2) + j(j+1)\right]\Phi_j \\ +(-1)^{j-1}2jc\Phi_{j-1} - j(j-1)\Phi_{j-2} = 0\end{aligned} \tag{45}$$

from which, using equation (44) and recurrence relations on Legendre polynomials, the Φ_j listed in Appendix II are obtained.

If equation (43) is substituted into equation (33), there results

$$_2P = \frac{\rho_a}{\rho_s}\frac{da/2}{h}\lambda W\frac{H}{B} \tag{46}$$

where H and B, infinite series in $1/a^2$ with coefficients that are functions of the Φ_j and c, may be expressed in the form

$$H = \sum_{k=0}^{\infty} a^{-2k}\{h_k[\Phi_j(c,\eta)] - icaq_k[c, \Phi_j(c,\eta)]\} \qquad j = 0, 1, 2, \ldots, 2k \tag{47}$$

$$B = \sum_{k=-1}^{\infty} a^{-2k} g_k[c, \Phi_j(c,\eta)], \qquad j = 0, 1, 2, \ldots, 2(k+1). \tag{48}$$

These expressions are not yet in an appropriate form since the coefficients of a^{-2k} are still functions of c.

The function Φ_j listed in Appendix II may be written as

$$\Phi_j(c,\eta) = \frac{1}{c^j}\sum_{k=0}^{j} c^{2k}\Phi_j^{(2k)}(\eta). \tag{49}$$

From equations (14) and (26)

$$(ca)^2 = b\lambda \tag{50}$$

where

$$b = \frac{\mu}{s^2\rho_s} \tag{51}$$

so that, from equation (39)

$$(ca)^2 = b\sum_{k=0}^{\infty} a^{-2k}\lambda^{(2k)} \tag{52}$$

from which expansions in powers of a^{-2k} can be obtained for $(ca)^{2m}$, for m integral or fractional, of the form

$$(ca)^{2m} = \sum_{k=0}^{\infty} a^{-2k} A_{mk}[\lambda^{(2j)}], \qquad j = 0, 1, 2, \ldots, k \tag{53}$$

where $A_{mk}[\lambda^{(2j)}]$ indicates that for each m, the coefficients of a^{-2k} are different functions of the indicated $\lambda^{(2j)}$. Substituting equations (49) and (53) into equations (47) and (48) one obtains

$$H = \sum_{k=0}^{\infty} a^{-2k} H^{(2k)} \tag{54}$$

and

$$B = \sum_{k=0}^{\infty} a^{-2k} B^{(2k)} \tag{55}$$

where the $H^{(2k)}$ and $B^{(2k)}$, not functions of a, are listed for $k = 0$ in Appendix III. (They have also been determined and used to obtain the results presented in this paper for $k = 1$–3, Ref. [5].)

If now the expansions of equations (38), (39), (54) and (55) are substituted into equation (46) the ${}_2P^{(2k)}$ and thus $\mathbf{P}^{(2k)}$ of equations (42) and (40) are obtained as

$${}_2P^{(2k)} = \sum_{j=0}^{k} \sum_{n=0}^{j} \lambda^{(2n)} \Lambda^{(2j-2n)} W^{(2k-2j)} \tag{56}$$

where the $\Lambda^{(2i)}$ are written out in Appendix IV and

$$r_0 = \frac{da}{2}. \tag{57}$$

It is important to point out that from equations (44) and (49) it follows that $H^{(0)}$ and $B^{(0)}$ are proportional to P_n^2 so that the $\Lambda^{(2i)}$ blow up at those values of η for which $P_n(\eta) = 0$. Therefore, although the ratio H/B and therefore the expression for ${}_2P$ of equation (46) has no singularities, the series obtained to express these are not convergent in the vicinity of the zeros of the Legendre polynomials.

Substituting equations (34), (36), (38), (39), (40) and (56) into equation (20) and equating like powers of a, an infinite number of differential equations on the $\mathbf{U}^{(2k)}$ are obtained as

$$\{\mathbf{L}^{(0)} - \lambda^{(0)}[\mathbf{M}^{(0)} + \overline{\mathbf{M}}^{(0)}\Lambda^{(0)}]\}\mathbf{U}^{(2i)} = \mathbf{F}^{(2i)} \tag{58}$$

where

$$\mathbf{F}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{59}$$

$$\begin{aligned} \mathbf{F}^{(2k)} = \begin{bmatrix} {}_1F^{(2k)} \\ {}_2F^{(2k)} \end{bmatrix} &= \sum_{j=1}^{k} [\lambda^{(2j)}\mathbf{M}^{(0)} + \lambda^{(2j-2)}\mathbf{M}^{(2)} - \mathbf{L}^{(2j)}]\mathbf{U}^{(2k-2j)} \\ &+ \sum_{j=1}^{k} \left\{ \overline{\mathbf{M}}^{(0)} \sum_{m=0}^{j} \lambda^{(2m)}\Lambda^{(2j-2m)} + \overline{\mathbf{M}}^{(2)} \sum_{m=0}^{j-1} \lambda^{(2m)}\Lambda^{(2j-2m-2)} \right\} \mathbf{U}^{(2k-2j)} \end{aligned} \tag{60}$$

with

$$\overline{\mathbf{M}}^{(2k)} = \begin{bmatrix} 0 & 0 \\ 0 & M^{(2k)} \end{bmatrix}, \qquad k = 0, 1. \tag{61}$$

4. DETERMINATION OF MODES AND FREQUENCIES

A. *Zero-order terms—vibrations of a submerged spherical shell*

Setting $i = 0$ in equation (58) and using equation (59) one obtains

$$[\mathbf{L}^{(0)} - \lambda^{(0)}(\mathbf{M}^{(0)} + \overline{\mathbf{M}}^{(0)}\Lambda^{(0)})]\mathbf{U}^{(0)} = 0 \quad (62)$$

which is identical to the limiting form of equation (20) as the spheroidal parameters approach those for spherical geometry, i.e.

$$a \to \infty, \qquad d \to 0, \quad \text{and} \quad \frac{da}{2} \to r_0. \quad (63)$$

Equation (62) is thus the equation governing the free vibration of a submerged spherical shell of radius r_0. Using the elements of the operator $\mathbf{L}^{(0)}$, $\mathbf{M}^{(0)}$ of equation (37), and $\Lambda^{(0)}$ listed in Appendix IV, equations (62) may be written as the two scalar equations

$$-(1-\eta^2)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}[(1-\eta^2)^{\frac{1}{2}}U^{(0)}] - (1-\nu)U^{(0)} - (1-\eta^2)^{\frac{1}{2}}(1+\nu)\frac{\mathrm{d}}{\mathrm{d}\eta}W^{(0)} - \lambda^{(0)}\frac{1-\nu}{2}U^{(0)} = 0 \quad (64)$$

$$(1+\nu)\frac{\mathrm{d}}{\mathrm{d}\eta}[(1-\eta^2)^{\frac{1}{2}}U^{(0)}] + 2(1+\nu)W^{(0)} - \frac{1-\nu}{2}\lambda^{(0)}W^{(0)}\left[1 + \frac{\rho_a}{\rho_s}\frac{r_0}{h}\frac{H^{(0)}}{B^{(0)}}\right] = 0. \quad (65)$$

These equations are satisfied by a solution of the same form as that used by Lamb [8] for the spherical shell *in vacuo*, i.e.

$$U_n^{(0)}(\eta) = (1-\eta^2)^{\frac{1}{2}}\frac{\mathrm{d}P_n}{\mathrm{d}\eta} \quad (66)$$

$$W_n^{(0)}(\eta) = K_n P_n \quad (67)$$

where n is the mode number, P_n is Legendre's function of the first kind and order n and

$$K_n = \frac{2n(n+1)\gamma}{4\gamma - \lambda_n^{(0)}\left[1 + \frac{\rho_a}{\rho_s}\frac{r_0}{h}\frac{H^{(0)}}{B^{(0)}}\right]} \quad (68)$$

with

$$\gamma = \frac{1+\nu}{1-\nu} \quad (69)$$

where $\lambda_n^{(0)}$ satisfies the frequency equation

$$\left[\lambda_n^{(0)}\left(1 + \frac{\rho_a}{\rho_s}\frac{r_0}{h}\frac{H^{(0)}}{B^{(0)}}\right) - 4\gamma\right][n(n+1)(\gamma+1) - (2+\lambda_n^{(0)})] + 4\gamma^2 n(n+1) = 0 \quad (70)$$

which (see expressions for $H^{(0)}$ and $B^{(0)}$ in Appendix III) yields only complex frequencies Ω with positive imaginary parts. The number of such roots depends on the mode number n.

Equation (70) may be shown to be identical to those obtained by Junger [9] and Mann [10]. Junger gave no numerical results for the spherical shell while Mann, who was not concerned *per se* with free vibrations, gives a response expression for one value of $(\rho_a/\rho_s)(r_0/h)$ whose denominator is a polynomial identical to the left-hand side of equation (70) for the same value of the cited parameter.

B. *Higher order terms*

The procedure followed is similar to that used in Ref. [2]. First $\lambda_n^{(2i)}$ is obtained once $\lambda_n^{(2j)}$ and $\mathbf{U}_n^{(2j)}$ for all $j < i$ are known. To accomplish this, both sides of equation (58) are premultiplied by $\tilde{\mathbf{U}}_n^{(0)}$ and integrated between $\eta = -1$ and $+1$ to give

$$\int_{-1}^{1} [\tilde{\mathbf{U}}^{(0)}\mathbf{L}^{(0)}\mathbf{U}^{(2i)} - \lambda^{(0)}\tilde{\mathbf{U}}^{(0)}(\mathbf{M}^{(0)} + \overline{\mathbf{M}}^{(0)}\Lambda^{(0)})\mathbf{U}^{(2i)}]\,\mathrm{d}\eta = \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\mathbf{F}^{(2i)}\,\mathrm{d}\eta \tag{71}$$

(where the mode subscript n is deleted in equation (71) and what follows) from which, making use of the self-adjointness of $\mathbf{L}^{(0)}$, $\mathbf{M}^{(0)}$, $\overline{\mathbf{M}}^{(0)}$, and equation (58), there follows

$$\int_{-1}^{+1} \tilde{\mathbf{U}}^{(0)}\mathbf{F}^{(2i)}\,\mathrm{d}\eta = 0. \tag{72}$$

If now equation (60) is substituted into equation (72), a non-transcendental solution for $\lambda^{(2i)}$ may be obtained as

$$\begin{aligned}\lambda^{(2i)} = {} & \frac{1}{\int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}[\mathbf{M}^{(0)} + \overline{\mathbf{M}}^{(0)}\Lambda^{(0)}(1-Q)]\mathbf{U}^{(0)}\,\mathrm{d}\eta} \times \Bigg\{ \sum_{j=1}^{i} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\mathbf{L}^{(2j)}\mathbf{U}^{(2i-2j)}\,\mathrm{d}\eta \\ & - \sum_{j=1}^{i-1} \lambda^{(2j)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\mathbf{M}^{(0)}\mathbf{U}^{(2i-2j)}\,\mathrm{d}\eta - \sum_{j=1}^{i} \lambda^{(2j-2)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\mathbf{M}^{(2)}\mathbf{U}^{(2i-2j)}\,\mathrm{d}\eta \\ & - \sum_{j=0}^{i-1} \lambda^{(2j)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\overline{\mathbf{M}}^{(0)}\Lambda^{(2i-2j)}\mathbf{U}^{(0)}\,\mathrm{d}\eta - \sum_{j=1}^{i-1}\sum_{l=0}^{j} \lambda^{(2l)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\overline{\mathbf{M}}^{(0)}\Lambda^{(2j-2l)}\mathbf{U}^{(2i-2j)}\,\mathrm{d}\eta \\ & - \sum_{j=1}^{i}\sum_{l=0}^{j-1} \lambda^{(2l)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\mathbf{M}^{(2)}\Lambda^{(2j-2l-2)}\mathbf{U}^{(2i-2j)}\,\mathrm{d}\eta - \lambda^{(2i)} \int_{-1}^{1} \tilde{\mathbf{U}}^{(0)}\overline{\mathbf{M}}^{(0)}\Lambda^{(0)}Q\mathbf{U}^{(0)}\,\mathrm{d}\eta \Bigg\}\end{aligned} \tag{73}$$

where the lengthy expression for Q is written out in Appendix VII of Ref. [5].

Once $\lambda^{(2j)}$ and the $\mathbf{U}^{(2j)}$ for all $j < i$ are known, $\mathbf{U}^{(2i)}$ can be determined. The two scalar equations corresponding to equation (58) are

$$-(1-\eta^2)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}[(1-\eta^2)^{\frac{1}{2}}U^{(2i)}] - \left[1 - \nu + \frac{1-\nu}{2}\lambda^{(0)}\right]U^{(2i)} - (1-\eta^2)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\eta}(1+\nu)W^{(2i)} = {}_1F^{(2i)} \tag{74}$$

$$(1+\nu)\frac{\mathrm{d}}{\mathrm{d}\eta}[(1-\eta^2)^{\frac{1}{2}}U^{(2i)}] + \left[2(1+\nu) - \lambda^{(0)}\frac{1-\nu}{2}\left(1 + \frac{\rho_a}{\rho_s}\frac{r_0}{h}\frac{H^{(0)}}{B^{(0)}}\right)\right]W^{(2i)} = {}_2F^{(2i)}. \tag{75}$$

Differentiating equation (75) with respect to η and substituting into (74) to eliminate $\mathrm{d}W^{(2i)}/\mathrm{d}\eta$, an equation on only $U^{(2i)}$ is obtained as

$$\begin{aligned} & -(1-\nu)\left(1 + \frac{\lambda^{(0)}}{2}\right)\left\{\frac{1}{n(n+1)}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}[(1-\eta^2)^{\frac{1}{2}}U^{(2i)}] + (1-\eta^2)^{-\frac{1}{2}}U^{(2i)}\right\} \\ & = (1-\eta^2)^{-\frac{1}{2}}\,{}_1F^{(2i)} + \frac{K_n}{n(n+1)}\frac{\mathrm{d}_2F^{(2i)}}{\mathrm{d}\eta}\end{aligned} \tag{76}$$

in which use has been made of equation (70).

The method of variation of the parameters is now applied to equation (76). Assuming

$$U^{(2i)} = U^{(0)}V(\eta) \tag{77}$$

where $U^{(0)}$ is given by equation (66) and $V(\eta)$ is an undetermined function, and making use of equation (70), $\mathrm{d}V/\mathrm{d}\eta$ is obtained by substituting equation (77) into equation (76). Integrating to get V and substituting into equation (77), $U^{(2i)}$ is obtained as

$$U_n^{(2i)} = (1-\eta^2)^{\frac{1}{2}}\frac{\mathrm{d}P_n}{\mathrm{d}\eta} \times \left\{1+\frac{n(n+1)}{K_n(1+\nu)-n(n+1)}\int\frac{\displaystyle\int\left[\frac{{}_1F_n^{(2i)}}{(1-\eta^2)^{\frac{1}{2}}}+\frac{K_n}{n(n+1)}\frac{\mathrm{d}\,{}_2F_n^{(2i)}}{\mathrm{d}\eta}\right](1-\eta^2)\frac{\mathrm{d}P_n}{\mathrm{d}\eta}\,\mathrm{d}\eta}{[(1-\eta^2)(\mathrm{d}P_n/\mathrm{d}\eta)]^2}\right\}\mathrm{d}\eta \tag{78}$$

Substituting equation (68) into (75), there results

$$W^{(2i)} = \frac{K_n}{n+1}\left[\frac{1}{1+\nu}\,{}_2F_n^{(2i)}-\frac{\mathrm{d}}{\mathrm{d}\eta}(1-\eta^2)^{\frac{1}{2}}U_n^{(2i)}\right] \tag{79}$$

which, in conjunction with (78) yields, after algebraic manipulation

$$W_n^{(2i)} = K_n\left\{P_n+\frac{n(n+1)}{K_n(1+\nu)-n(n+1)}\int\frac{\mathrm{d}P_n}{\mathrm{d}\eta} \times\left[\int\frac{\displaystyle\int\left[\frac{{}_1F_n^{(2i)}}{(1-\eta^2)^{\frac{1}{2}}}+\frac{K_n}{n(n+1)}\frac{\mathrm{d}\,{}_2F_n^{(2i)}}{\mathrm{d}n}\right](1-\eta^2)\frac{\mathrm{d}P_n}{\mathrm{d}\eta}\,\mathrm{d}\eta}{[(1-\eta^2)(\mathrm{d}P_n/\mathrm{d}\eta)]^2}\,\mathrm{d}\eta\right]\mathrm{d}\eta\right\} - \frac{K_n}{K_n(1+\nu)-n(n+1)}\left[\int\frac{{}_1F_n^{(2i)}}{(1-\eta^2)^{\frac{1}{2}}}\,\mathrm{d}\eta+\frac{{}_2F_n^{(2i)}}{1+\nu}\right] \tag{80}$$

where the constants of integration are determined by requiring that equations (78) and (80) reduce to equations (66) and (67) for $i = 0$.

5. INCOMPRESSIBLE FLUID

If the velocity of sound in the fluid is made to approach infinity then, from equations (26) and (51)

$$c \to 0. \tag{81}$$

$$b \to 0. \tag{82}$$

Since the series for $H^{(2i)}$ and $B^{(2i)}$, given in Appendix III, have a finite number of terms for each value of the mode number n, and the highest power of $1/b\lambda^{(0)}$ is the same for all values of i for any given n, the ratios $H^{(2i)}/H^{(0)}$ and $B^{(2i)}/B^{(0)}$ approach a finite limit as $b \to 0$. In this limit the imaginary parts of the $H^{(2i)}$, and therefore of the $\Lambda^{(2i)}$ and Q vanish. Therefore, the damping due to the fluid vanishes and its only effect on the vibration of the shell is the addition of a "virtual mass" of fluid to the mass of the shell *in vacuo* for radial motion, leading to real frequencies which, while evidently lower than these for the shell *in vacuo*, are found to be upper bounds to the real part of the fundamental frequency in a compressible fluid.

6. ACOUSTIC IMPEDANCE

The acoustic impedance ratio ζ may be defined as

$$\zeta = \frac{p_a}{\rho_a s \dot{w}} \tag{83}$$

where p_a and $\dot{w}$ are the fluid pressure on the shell surface and radial velocity there due to a forced harmonic excitation of real frequency ω.

Using equations (10), (29), (31) and (50) and the definition of H/B from equations (33) and (46), with ω replacing the natural frequency Ω, equation (83) may be written as

$$\zeta = i\sqrt{(b\lambda)}\bigg/\sqrt{\left(\frac{a^2-\eta^2}{a^2-1}\right)}\frac{H}{B} \tag{84}$$

where λ, obtained from equation (14) with ω replacing Ω, is real and continuously variable.

By decomposing the right-hand side of equation (84) into real and imaginary parts the acoustic impedance may be written as

$$\zeta = \theta + i\chi \tag{85}$$

where θ and χ are the acoustic resistance and reactance, respectively.

Since the expressions for H and B are in terms of the $\lambda^{(2i)}$ and it is desirable to know θ and χ as functions of η and $\sqrt{(b\lambda)}$ it was found expedient to choose

$$\lambda^{(2i)} = q \tag{86}$$

so that continuous variation of λ corresponding to continuous variation of q follows by substituting equation (86) into equation (39) and adding the geometric series:

$$\lambda = q\left(\frac{1}{1-1/a^2}\right). \tag{87}$$

7. NUMERICAL RESULTS FOR THE MODE $n = 2$ OF STEEL SHELLS IN WATER

For steel shells in water using the following material properties as representative

$$E = 29 \times 10^6 \text{ lb/in}^2$$

$$\nu = 0{\cdot}3$$

$$\rho_s = 0{\cdot}2836 \text{ lb/in}^3$$

$$\rho_a = 0{\cdot}0370 \text{ lb/in}^3$$

$$s = 4975 \text{ ft/sec}$$

the terms $\lambda_n^{(2i)}$ and $U_n^{(2i)}$ of the series for frequency parameters and mode shapes of equations (39) and (38) were computed for $n = 2$ and various ratios r_0/h of shell length to thickness from equations (73), (78) and (80).

For $n = 2$, the frequency equation (70) for spherical geometry yields 7 complex roots $\sqrt{(\lambda_2^{(0)})}$ (corresponding to seven complex frequencies by means of equation (14)) with positive imaginary parts (one with zero real part) shown, for various ratios r_0/h, in Table 1.

TABLE 1. $\sqrt{\lambda_2^{(0)}}$

j, root identifying number \ $\frac{r_0}{h}$	25	50	100	200
1, 2	$\pm 0{\cdot}80885026$ $+0{\cdot}047701064i$	$\pm 0{\cdot}67806300$ $+0{\cdot}024622887i$	$\pm 0{\cdot}54633071$ $+0{\cdot}0094567567i$	$\pm 0{\cdot}42250395$ $+0{\cdot}0026401789i$
3, 4	$\pm 4{\cdot}5101644$ $+0{\cdot}21698105i$	$\pm 4{\cdot}3424075$ $+0{\cdot}33118187i$	$\pm 4{\cdot}0935453$ $+0{\cdot}30745044i$	$\pm 3{\cdot}9610749$ $+0{\cdot}18608506i$
5, 6	$\pm 1{\cdot}1558071$ $+1{\cdot}0133781i$	$\pm 0{\cdot}55392193$ $+1{\cdot}5505173i$	$\pm 0{\cdot}40568554$ $+0{\cdot}94320466i$	$\pm 0{\cdot}40955821$ $+0{\cdot}83518950i$
7	$+0{\cdot}96234118i$	$+1{\cdot}2858262i$	$+5{\cdot}7382642i$	$+12{\cdot}530693i$

Of these, the two pairs with low damping ratios approach the real roots for vibration *in vacuo* as the fluid density diminishes.

In Table 2, all seven values of $\lambda_2^{(2i)}$ are shown for $i = 0, 1, 2$ and 3 for $r_0/h = 25$.

Letting

$$\sqrt{\lambda} = \alpha + i\beta \tag{88}$$

a damping ratio τ may be defined by

$$\tau = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}. \tag{89}$$

TABLE 2. $\lambda_2^{(2i)}$ FOR $r_0/h = 25$

i \ j	1, 2	3, 4	5, 6	7
0	$0{\cdot}65196335 \pm 0{\cdot}077166037i$	$20{\cdot}294502 \pm 1{\cdot}9572404i$	$0{\cdot}30895482 \pm 2{\cdot}3425393i$	$-0{\cdot}92610054$
1	$1{\cdot}1306226 \pm 0{\cdot}30451567i$	$12{\cdot}013223 \pm 1{\cdot}1699525i$	$1{\cdot}4968430 \pm 1{\cdot}0584341i$	$-0{\cdot}12907169$
2	$1{\cdot}309207 \pm 0{\cdot}9016935i$	$10{\cdot}35547 \pm 2{\cdot}516328i$	$3{\cdot}712397 \pm 0{\cdot}8148246i$	$-4{\cdot}461715$
3	$-3{\cdot}01765 \pm 0{\cdot}0845010i$	$6{\cdot}52813 \pm 6{\cdot}73996i$	$1{\cdot}45802 \pm 1{\cdot}14172i$	-61.8747

Close examination of Tables 1 and 2 shows that the convergence of the series for λ becomes poor when $a^2 < 3$. Plots of α, β and τ as functions of r_0/h are given in Figs. 3, 4 and 5.

In Table 3 similar numerical results for an incompressible fluid are presented.

Finally, in Figs. 6 and 7, plots of the impedances χ and θ are given for $\eta = 0, \pm 1$, and $\pm 1/\sqrt{3}$ for a spheroidal shell with $a^2 = 3$, and compared with the η-independent values for a spherical shell ($a^2 = \infty$). It should be noted that for $\eta = \pm 1$, the spheroidal values for $a^2 = 3$ do not differ enough from the spherical values to be able to distinguish them on a plot.

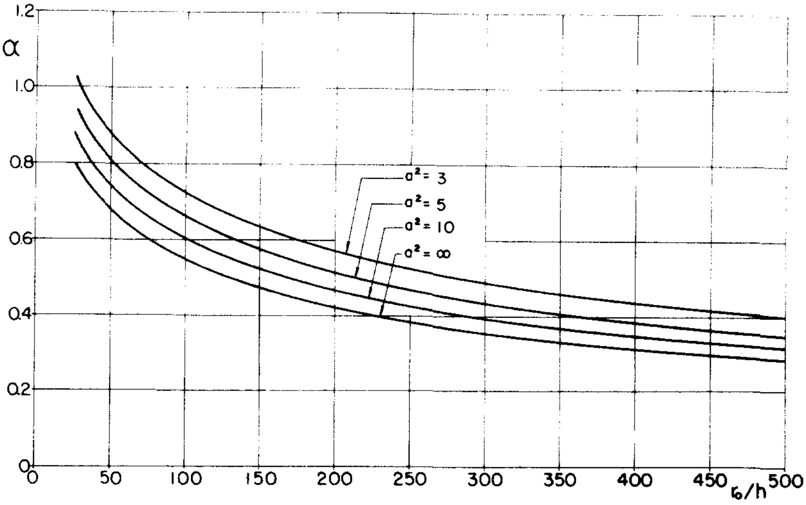

FIG. 3. Damped frequency α, $\sqrt{\lambda} = \alpha + i\beta$.

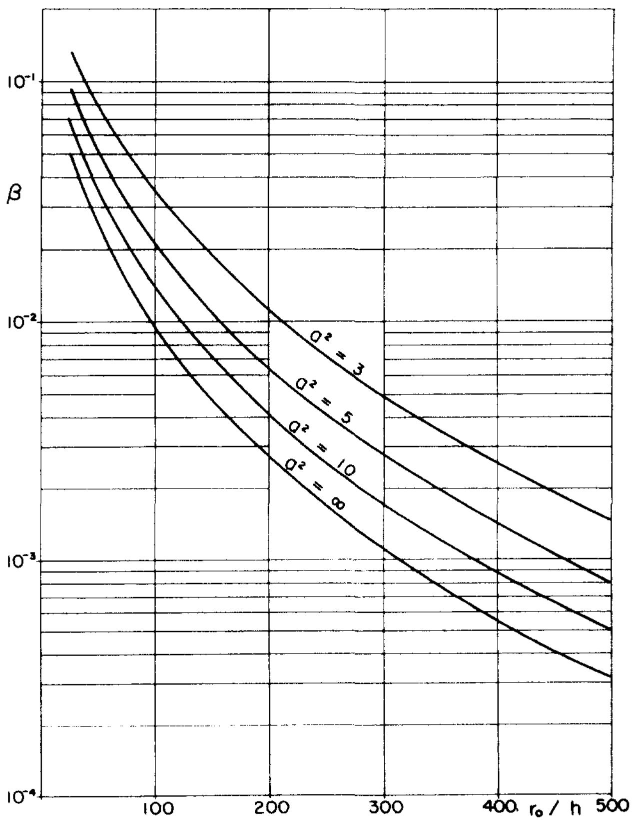

FIG. 4. Damping factor β, $\sqrt{\lambda} = \alpha + i\beta$.

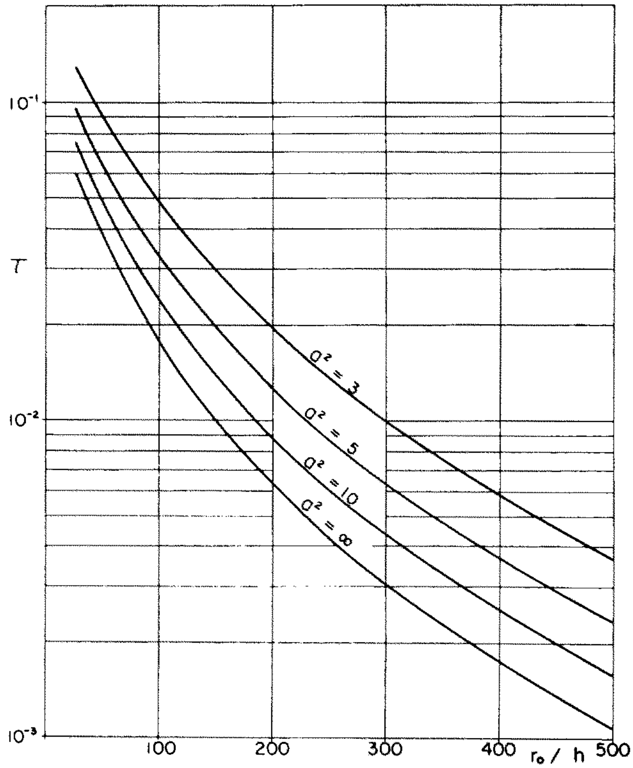

FIG. 5. Damping ratio $\tau = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}$.

TABLE 3. FUNDAMENTAL VALUES OF $\lambda_2^{(2i)}$, INCOMPRESSIBLE FLUID

i	$r_0/h = 25$	$r_0/h = 50$
0	0·79515627	0·55302585
1	1·3722756	1·0079289
2	1·530256	1·151258
3	1·15229	0·818057

8. CONCLUSION

The perturbation technique presented herein has led to the first determination of complex frequencies of submerged spheroidal shells, but it is too cumbersome to apply to shells of larger eccentricity, when the convergence is very slow, to be practically useful. The numerical results obtained, however, should provide valuable check points for any approximate or numerical solutions which may be formulated.

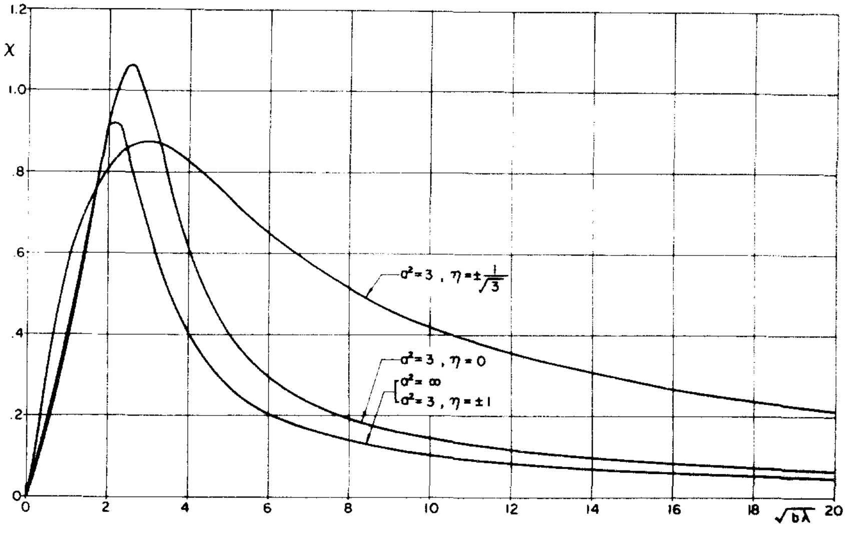

FIG. 6. Acoustic reactance $\chi_n(\sqrt{b\lambda}, \eta)$, $n = 2$.

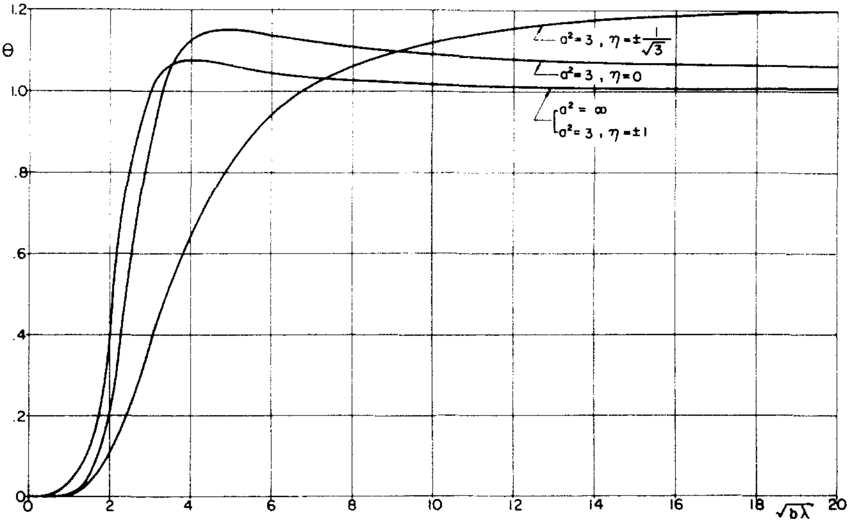

FIG. 7. Acoustic resistance $\theta_n(\sqrt{b\lambda}, \eta)$, $n = 2$.

Acknowledgments—This work was supported by the Office of Naval Research Contract Nonr 266(67).

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APPENDIX I. SERIES EXPANSION OF FLUID POTENTIAL

The prolate spheroidal radial functions R_{on} satisfy the differential equation [7].

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(\xi^2-1)\frac{\mathrm{d}R_{on}}{\mathrm{d}\xi}\right]-(l_{on}-c^2\xi^2)R_{on} = 0 \tag{I-a}$$

where the l_{on} are eigenvalues. This has regular singular points at $\xi = \pm 1$ and an irregular singular point of rank one at ∞. Letting

$$R_{on}(c,\xi) = (\xi^2-1)^{-\frac{1}{2}}y(c,\xi) \tag{I-b}$$

equation (a) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+\left[\frac{1}{(\xi^2-1)^2}-\frac{l_{on}-c^2\xi^2}{\xi^2-1}\right]y = 0. \tag{I-c}$$

Expanding the bracketed term into a series of even negative powers of ξ, i.e.

$$\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+y\sum_{k=0}^{\infty}\alpha_k\xi^{-2k} = 0 \tag{I-d}$$

and substituting into (d) an assumed formal solution of the form [7]

$$y = \mathrm{e}^{\sigma\xi}\sum_{v=0}^{\infty}a_v\xi^{-v} \tag{I-e}$$

yields, upon equating like powers of ξ

$$\sigma = \pm ic \tag{I-f}$$

and R_{on} is obtained as a series of the form

$$R_{on}(c,\xi) = \frac{\mathrm{e}^{\pm ic\xi}}{c\sqrt{\xi^2-1}}\left\{\left[\bar{a}_0+\frac{\bar{a}_2}{\xi^2}+\frac{\bar{a}_4}{\xi^4}+\cdots\right]\pm i\left[\frac{\bar{a}_1}{\xi}+\frac{\bar{a}_3}{\xi^3}+\cdots\right]\right\} \tag{I-g}$$

where all the $\bar{a}_k$ are related to $\bar{a}_0$ by recurrence relations and the plus and minus signs signify, respectively, incoming and outgoing waves. Utilizing the form of R_{on} obtained in (g), one is led to try a similar series for Φ of equation (25) with functions of η replacing the constants. Thus, for outgoing waves and $\xi > 1$, a series of the form of equation (43) results.

If the appropriate limits of equation (63) are taken to reduce (g) to spherical geometry, the spherical Hankel functions of the first and second kind are obtained. Applying the same limiting process to equation (43), it will be seen that, to reduce to the known [12] spherical solution of the scalar wave equation, equation (44) must be satisfied.

APPENDIX II. Φ_j

$$\Phi_0 = P_n$$

$$\Phi_1 = \frac{1}{2c}\{n(n+1)P_n - c^2(1-\eta^2)P_n\}$$

$$\Phi_2 = \frac{1}{2^2 2! c^2}\left\{-(n-1)n(n+1)(n+2)P_n + c^2(1-\eta^2)\left[2n(n-1)P_n + 4\frac{dP_{n+1}}{d\eta}\right] - c^4(1-\eta^2)^2 P_n\right\}$$

$$\Phi_3 = \frac{1}{2^3 3! c^3}\left\{-(n-2)(n-1)n(n+1)(n+2)(n+3)P_n\right.$$
$$+c^2(1-\eta^2)\left[3(n-2)(n-1)n(n+1)P_n + 12n(n+1)\frac{dP_{n+1}}{d\eta}\right]$$
$$\left.-c^4(1-\eta^2)^2\left[3(n-2)(n-1)P_n + 12\frac{dP_{n+1}}{d\eta}\right] + c^6(1-\eta^2)^3 P_n\right\}$$

$$\Phi_4 = \frac{1}{2^4 4! c^4}\left\{(n-3)(n-2)(n-1)\ldots(n+2)(n+3)(n+4)P_n\right.$$
$$-c^2(1-\eta^2)\left[4(n-3)(n-2)(n-1)n(n+1)(n+2)P_n + 24(n-1)n(n+1)(n+2)\frac{dP_{n+1}}{d\eta}\right]$$
$$+c^4(1-\eta^2)^2\left[6(n-3)(n-2)(n-1)nP_n + 48(n-1)n\frac{dP_{n+1}}{d\eta} + 48\frac{d^2P_{n+2}}{d\eta^2}\right]$$
$$\left.-c^6(1-\eta^2)^3\left[4(n-3)(n-2)P_n + 24\frac{dP_{n+1}}{d\eta}\right] + c^8(1-\eta^2)^4 P_n\right\}.$$

A more extensive list of Φ_i up to $i = 7$ is given in Ref. [5].

APPENDIX III. $H^{(2i)}$ AND $B^{(2i)}$

$$H^{(0)} = \left\{\Phi_0^{(0)^2} + \frac{2}{(b\lambda^{(0)})}[\Phi_1^{(0)^2} + 2\Phi_0^{(0)}\Phi_2^{(0)}] + \frac{3}{(b\lambda^{(0)})^2}[\Phi_2^{(0)^2} + 2\Phi_0^{(0)}\Phi_4^{(0)} + 2\Phi_1^{(0)}\Phi_3^{(0)}]\right.$$
$$\left.+\frac{4}{(b\lambda^{(0)})^3}[\Phi_3^{(0)^2} + 2\Phi_0^{(0)}\Phi_6^{(0)} + 2\Phi_1^{(0)}\Phi_5^{(0)} + 2\Phi_2^{(0)}\Phi_4^{(0)}] + \ldots\right\}$$
$$-i\sqrt{b\lambda^{(0)}}\left\{\Phi_0^{(0)^2} + \frac{1}{(b\lambda^{(0)})}[\Phi_1^{(0)^2} + 2\Phi_0^{(0)}\Phi_2^{(0)} - \Phi_0^{(0)}\Phi_1^{(0)}]\right.$$

$$+\frac{1}{(b\lambda^{(0)})^2}[\Phi_2^{(0)^2}+2\Phi_0^{(0)}\Phi_4^{(0)}+2\Phi_1^{(0)}\Phi_3^{(0)}-3\Phi_0^{(0)}\Phi_3^{(0)}+\Phi_1^{(0)}\Phi_2^{(0)}]$$

$$+\frac{1}{(b\lambda^{(0)})^3}[\Phi_3^{(0)^2}+2\Phi_0^{(0)}\Phi_6^{(0)}+2\Phi_1^{(0)}\Phi_5^{(0)}+2\Phi_2^{(0)}\Phi_4^{(0)}-5\Phi_0^{(0)}\Phi_5^{(0)}$$

$$\left.+3\Phi_1^{(0)}\Phi_4^{(0)}-\Phi_2^{(0)}\Phi_3^{(0)}]+\ldots\right\}$$

$$B^{(0)}=(b\lambda^{(0)})\Phi_0^{(0)^2}+[\Phi_0^{(0)^2}-2\Phi_0^{(0)}\Phi_1^{(0)}+\Phi_1^{(0)^2}+2\Phi_0^{(0)}\Phi_2^{(0)}]$$

$$+\frac{1}{(b\lambda^{(0)})}[4\Phi_1^{(0)^2}+6\Phi_0^{(0)}\Phi_2^{(0)}-6\Phi_0^{(0)}\Phi_3^{(0)}+2\Phi_1^{(0)}\Phi_2^{(0)}+\Phi_2^{(0)^2}+2\Phi_0^{(0)}\Phi_4^{(0)}+2\Phi_1^{(0)}\Phi_3^{(0)}]$$

$$+\frac{1}{(b\lambda^{(0)})^2}[9\Phi_2^{(0)^2}+10\Phi_0^{(0)}\Phi_4^{(0)}+16\Phi_1^{(0)}\Phi_3^{(0)}-10\Phi_0^{(0)}\Phi_5^{(0)}$$

$$+6\Phi_1^{(0)}\Phi_4^{(0)}-2\Phi_2^{(0)}\Phi_3^{(0)}+\Phi_3^{(0)^2}+2\Phi_0^{(0)}\Phi_6^{(0)}+2\Phi_1^{(0)}\Phi_5^{(0)}+2\Phi_2^{(0)}\Phi_4^{(0)}]$$

$$+\frac{1}{(b\lambda^{(0)})^3}[16\Phi_3^{(0)^2}+14\Phi_0^{(0)}\Phi_6^{(0)}+24\Phi_1^{(0)}\Phi_5^{(0)}+30\Phi_2^{(0)}\Phi_4^{(0)}-14\Phi_0^{(0)}\Phi_7^{(0)}+10\Phi_1^{(0)}\Phi_6^{(0)}$$

$$-6\Phi_2^{(0)}\Phi_5^{(0)}+2\Phi_3^{(0)}\Phi_4^{(0)}+\Phi_4^{(0)^2}+2\Phi_0^{(0)}\Phi_8^{(0)}+2\Phi_1^{(0)}\Phi_7^{(0)}+2\Phi_2^{(0)}\Phi_6^{(0)}+2\Phi_3^{(0)}\Phi_5^{(0)}]$$

$$+\ldots.$$

For any finite value of the mode number n, $H^{(2i)}$ and $B^{(2i)}$ contain only a finite number of terms. For $n = 2$, e.g. all terms multiplied by a factor with powers of $1/b\lambda^{(0)}$ larger than or equal to 3 are identically zero.

A more extensive list of $H^{(2i)}$ and $B^{(2i)}$ may be found in Ref. [5].

APPENDIX IV. $\Lambda^{(2j)}$

$$\Lambda^{(0)}=\left(\frac{\rho_a}{\rho_s}\frac{r_0}{h}\right)\frac{H^{(0)}}{B^{(0)}}$$

$$\Lambda^{(2)}=\left(\frac{\rho_a}{\rho_s}\frac{r_0}{h}\right)\frac{H^{(0)}}{B^{(0)}}\left\{\frac{H^{(2)}}{H^{(0)}}-\frac{B^{(2)}}{B^{(0)}}\right\}$$

$$\Lambda^{(4)}=\left(\frac{\rho_a}{\rho_s}\frac{r_0}{h}\right)\frac{H^{(0)}}{B^{(0)}}\left\{\frac{H^{(4)}}{H^{(0)}}-\frac{B^{(2)}}{B^{(0)}}\frac{H^{(2)}}{H^{(0)}}+\left(\frac{B^{(2)}}{B^{(0)}}\right)^2-\frac{B^{(4)}}{B^{(0)}}\right\}.$$

(*Received* 31 *January* 1969; *revised* 14 *July* 1969)

Абстракт—Определяются дифференциальные уравнения, касающиеся осесимметрических колебаний удлинения упругой удлиненной сфеоидальной оболочки, погруженной в бесконечной акустической среде, в выражениях удлиненных сфероидальных координат, используя закон Гамильтона.

Решения получаются методом возмущения, обладающим надлежащей сходимостью для оболочек с малым эксцентрицитетом.

Даются численные результаты для основного вида колебаний, частоты и акустического импеданса для стальных оболочек в воде, при отношениях большой оси к меньшей до 1,23 и для широкого круга отношений длины оболочки к толщине.